Section 2.5

Infinite limits are characterized by function values steadily growing to arbitrary large sizes as the variable x approaches a value c. In this section we study how functions behave as the variable is increased or decreased to arbitrarily large magnitudes.

Limits at Infinity

Definition 2.5.1. If $f(x)$ becomes arbitrarily close to a number L for all sufficiently large and positive x we write

$$\lim_{x\to\infty} f(x) = L.$$

We say "the limit of f as x approaches infinity is L." And note that this limit exists.
If $f(x)$ becomes arbitrarily close to a number L for all sufficiently large and negative x we write

$$\lim_{x\to-\infty} f(x) = L.$$

We say "the limit of f as x approaches minus infinity is L."

In either case, the line $y = L$ is a *horizontal asymptote* of f.

For limits at infinity, all reasonably applicable limit laws apply.

Example 2.5.2. Evaluate the limits:

1. $\lim_{x\to\infty} \frac{10}{x^2}$

2. $\lim_{x\to\infty} \frac{C}{x^n}$

3. $\lim_{x \to \infty} \frac{\sin x}{\sqrt{x}}$

4. $\lim_{x \to -\infty} \frac{x}{x+2}$

5. $\lim_{x \to \infty} \frac{3x^3 - 2x^2 - 5x + 10}{4x^3 + 6x^2 + 4}$

Limits at infinity can be themselves infinite, and the definition is what we expect. In general, the evaluation of limits and infinity is called *end behavior*.

Definition 2.5.3. If $f(x)$ becomes arbitrarily large as the variable x becomes arbitrarily large, then we write

$$\lim_{x\to\infty} f(x) = \infty.$$

The three limits $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$ are defined analogously. *These limits do not exist.*

Example 2.5.4. Compute

1. $\lim_{x\to\infty} 3x^4 - 5x^2 - 7x + 3$

2. $\lim_{x\to\infty} 4x^3 - 2x^2 - x - 5$

3. $\lim_{x\to\infty} x^n$

4. $\lim_{x \to -\infty} x^n$

5. $\lim_{x \to -\infty} \sin x$

Example 2.5.5.

1. $\lim_{x \to \infty} \dfrac{10x^3 + 3x^2 - 4x + 5}{8x^4 + 3x^2 - 9x + 1}$

2. $\lim_{x\to\infty} \dfrac{x^3 - 2x + 1}{2x + 4}$

Theorem 2.5.6 (End Behavior of Rational Functions)**.** Suppose

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$$

with $a_n \neq 0$ and $b_m \neq 0$ and let $f(x) = \dfrac{p(x)}{q(x)}$.

1. $\mathbf{n < m}$**:** $\lim_{x\to\pm\infty} f(x) = 0$ and $y = 0$ is a horizontal asymptote of f.
2. $\mathbf{n = m}$**:** $\lim_{x\to\pm\infty} f(x) = \dfrac{a_n}{b_m}$ and $y = \frac{a_n}{b_m}$ is a horizontal asymptote of f.
3. $\mathbf{m < n}$**:** $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$ and f has no horizontal asymptote.
4. $\mathbf{n = m + 1}$**:** $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$ and f has no horizontal asymptote, but has a slant asymptote.

Example 2.5.7. Find

1. $\lim_{x\to\infty} \dfrac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$

2. $\lim_{x\to-\infty} \dfrac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$

Theorem 2.5.8 (End Behavior of e^x, e^{-x}, and $\ln x$)**.** End behavior is as follows:

$$\lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to-\infty} e^x = 0$$
$$\lim_{x\to\infty} e^{-x} = 0 \qquad \lim_{x\to-\infty} e^{-x} = \infty$$
$$\lim_{x\to0^+} \ln x = -\infty \qquad \lim_{x\to\infty} \ln x = \infty$$

Example 2.5.9. Find

1. $\lim_{x\to\infty} 3e^{2x}$ $\qquad\qquad$ $\lim_{x\to-\infty} 3e^{2x}$

2. $\lim_{x\to\infty} 4 - \ln 3x$